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Geometric bounds on Kaluza–Klein masses

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Abstract. We point out geometric upper and lower bounds on the masses of bosonic and fermionic Kaluza–Klein excitations in the context of theories with large extra dimensions. The characteristic compactification length scale is set by the diameter of the internal manifold. Based on geometrical and topological considerations, we find that certain choices of compactification manifolds are more favored for phenomenological purposes.

1 Introduction and summary

In the recent past, there has been paid much attention to models with large extra dimensions. The surge in activity surrounding this idea owes its origin to the belief that the existence of extra dimensions (beyond four) seems to be a crucial ingredient for the unification of gravity with gauge forces. The initial goal of taking a large radius, $r \gg M_{\mathrm{P}}^{-1}$, in compactification schemes is to weaken the hierarchy between the electroweak scale and the four-dimensional gravity scale, M_{P}. The idea is that the matter content of the standard model of elementary particles (SM) is confined to $(3+1)$ dimensions, as suggested by [1–4], while gravity lives in the whole D-dimensional space ($D > 4$). Upon compactification, the hierarchy problem is solved by lowering the fundamental scale of gravity, M_*, down to TeV through a model-dependent relation between M_* and M_{P}. The compactification mechanisms suggested so far can be classified into two broad categories: models with the tensor product of our four-dimensional world with the internal space [2,3] (in line with the original Kaluza–Klein ideology), and models with a warp product [5,6] between these spaces.

On compactifying down to four dimensions, one may in general get new degrees of freedom added to the SM spectrum. The new states can be purely from the gravitational sector, or have standard model KK excitations in addition (depending on whether the SM interactions are written directly in four dimensions, using the induced metric, or written fully in D dimensions). In any case, the new states might lead to detectable modifications of the existing accelerator data and cosmological observations [9]. The phenomenology of both categories has been much investigated during the past two years, and we refer for a summary of the recent findings to [7]. The usual way to avoid such new contributions to the prevailing scenario at low energies is often either by decoupling the particles by making their masses very heavy (beyond the present reach of accelerators, say $\gtrsim$ TeV), or by imposing judicious bounds on their couplings and masses. Recently, it was suggested [8] that the heavy masses could be realized naturally (without fine-tuning), utilizing only certain geometrical properties of the internal manifold, namely that the masses arising from compactification are exponentially large, being related to the volume of the internal hyperbolic manifold.

In this work we scrutinize the criteria for choosing the internal manifold, in both the case of tensor and of warp product compactifications, based on geometrical and topological arguments, such that the unwanted KK contributions are avoided. We focus on compact, connected, and smooth internal manifolds with scalar curvature, bounded from below $\kappa \geq (d-1)K$, where K is a constant. We consider, for generality, a gravity theory coupled to a Dirac spinor in the presence of a gauge theory. This consideration has the final aim of being applied to the SM. However, in order to keep the discussion simple and sufficiently general (model independent) we shall not concern ourselves with finer details like localization mechanisms, the issue of obtaining chiral fermions starting from odd dimensions, etc. Instead, after performing the general analysis of various bounds on the KK masses, one may specialize to the case of the SM.

Most of our analysis relies on the following basic observations concerning the spectrum of Riemannian manifolds, and the Dirac operator on spin manifolds. The main fact is that the spectrum of $\not{D}$ and that of the Laplacian on compact Riemmanian manifolds is discrete, bounded from below, and the eigenvalues (counted with multiplicity) are ordered: $0 = \lambda_0 \leq \lambda_n \leq \lambda_{n+1}$. Moreover, there exist lower bounds on λ_1 of a Laplacian acting on a scalar in the compact manifold. In addition, there are upper bounds

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on the eigenvalues which sets a ceiling to how heavy they can become. These translate into lower bounds on the four-dimensional tree-level masses of particles arising from compactification. For spinors, the classic theorem of Lichnerowicz enables us to impose similar bounds, upper and lower, and altogether exclude tree-level massless fermions for certain internal manifolds. To sum up, we use topological considerations to comment on bosonic KK zero modes, while we use geometrical arguments to impose bounds on fermions and massive KK modes.

This paper is organized as follows: in Sect. 2, we summarize our conventions and state our requirements for choosing the internal manifold so that we are able to produce a phenomenologically reliable scenario in a rather model-independent way. The implementation of these demands is carried out in the successive sections. In Sect. 3, we relate the eigenvalues of the Laplacian on the internal space with the tree-level masses in four dimensions, and discuss geometrical upper and lower bounds for massive bosons, and topological conditions for the massless ones. In Sect. 4, we comment on massless spinors using the Lichnerowicz theorem, and point out the existence of curvature-dependent upper and lower bounds on the massive ones. Demanding the satisfaction of all of our requirements stated in the second section, we are able to rule out certain choices of the compactification manifolds. Finally, we summarize our conclusions in Sect. 5.

2 Conventions and set-up

As mentioned in the introduction, we consider Einstein's gravity coupled to a Dirac spinor and a Yang–Mills gauge theory on a D-dimensional manifold $W = M_4 \cup Y$,

$$S = \int_W \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{G}\mathcal{R} + \frac{1}{4}F^2 + \mathrm{i}\bar{\psi}\hat{\not{D}}_A\psi \right]$$

where M_4 is a four-manifold, which we eventually identify as our four-dimensional world, and Y is a compact $(D-4)$-dimensional manifold. $\hat{\not{D}}_A$ is the twisted Dirac operator on W and F is the YM field strength (for a detailed treatment of an analogous setting in six and ten dimensions, and for the conventions, we refer to [10]). Consequently, the various fields on W will be decomposed as follows: scalars on W will be scalars on both M_4 and Y; a vector on W will be a vector on M_4 and a scalar on Y or vice versa; the graviton on W will appear as a graviton on M_4 and a scalar on Y or vice versa, or as a vector on both submanifolds. Finally, a spinor on the parent manifold will decompose as a spinor on both M_4 and Y. It is perhaps worth mentioning that a spinor defined on W which is a fibre product of M_4 and Y does not necessarily split into spinors defined on the two submanifolds individually, as it does in the Cartesian tensor product case. However, in the special case of a warp product, the fibration being trivial, this decomposition once again holds.

Our analysis includes the tensor product decomposition (the standard Kaluza–Klein compactification), with aside comments about it and the warp product [17] decomposition. Let us recall that for the tensor product, $W = M_4 \otimes Y$, the inherited metric is $\hat{g} = g_4 + g_Y$ where g_4 and g_Y are the metrics on M_4 and Y, respectively. Whereas for a warp product $W = M_4 \otimes_{\mathbb{R}^+} Y$, the resultant inherited metric is of the form $\hat{g} = f^2 g_4 + g_Y$, where f is a smooth map $f : Y \to \mathbb{R}^+$. In this work we choose the warp factor to be $f = \mathrm{e}^{-(1/2)\phi}$ as in [6]. The warp factor is to be consistently determined by solving Einstein's equations.

Our main requirements for the theory resulting after compactification can be outlined as follows:

(1) With respect to the gravity sector, we want to end up with one massless graviton, no additional massless gauge bosons, and no massless scalars[1].
(2) The zeroth KK mode(s) of the Dirac spinor is massless in four dimensions. If specialized to the SM, this translates into the requirement that fermion masses are exclusively due to the Higgs mechanism.
(3) The masses of the KK excitations of various fields are naturally heavy.

3 Bounds on bosonic KK masses

The starting point of our analysis is the examination of the masses in the gravity sector. Looking at the linearized Einstein equations on W[2], $\hat{\Delta} h_{\bar{k}\bar{l}} = T_{\bar{k}\bar{l}}$ (the $\bar{k}$'s are the indices on W), one can relate the spectrum of Δ_Y with the tree-level masses of the various fields on M_4.

The linearization approach is usually performed assuming a weak gravitational field. However, in the spirit of large radii compactification scenarios, the scalar curvature is expected to be in units of the compactification radius, and hence of order TeV^2. Therefore, neglecting the curvature may not always be justified. Having said this, we will here stick to the conventional linearization, this being the only simple way at hand to deal with gravity.

The parent Laplacian[3], $\hat{\Delta}$, decomposes as

$$\hat{\Delta} = \Delta_4 + \Delta_Y \tag{1}$$

in the tensor product case, and as

[1] Massless vectors may enhance the gauge symmetry, and gravitational interactions mediated by scalars violate the equivalence principle

[2] We use the linearized equations for obvious reasons; nevertheless, we include for completeness the decomposition of $\mathcal{R}$ on M_4 and Y,

$$\mathcal{R} = \mathcal{R}_4 + \kappa$$

for the tensor product; and for the warp product [17]

$$\mathcal{R} = \frac{1}{f^2}\left\{\mathcal{R}_4 - 8f\Delta f - 6 \parallel \nabla f \parallel^2\right\} + \kappa,$$

where f is the warp factor, which we may assume to be $f = \mathrm{e}^{-(1/2)\phi}$

[3] The "hat" superscript refers to quantities defined on W

$$\hat{\Delta} = e^{\phi(y)} \Delta_4 + \Delta_Y - \frac{1}{2} \left(\partial_l \phi(y) \right) \partial^l \qquad (2)$$

in the warp product case (the l's are the indices on Y).

3.1 Massless bosons

A necessary condition to meet the first demand (1) is to select Y with the appropriate Betti numbers[4]. Since $b_0(Y) = 1$ for any general connected manifold Y, we are guaranteed to end up with one massless graviton on M_4. $b_1(Y) = 0$ would ensure that no new massless vector bosons, nor massless scalars are produced in M_4 after compactification. However, this is not the case for a general Y. For example, a circle has $b_1(S^1) = 1$ and for a torus, T^d, $b_1 = d$, both of which therefore admit massless 1-forms. S^d's are in general suitable ambient spaces for performing such compactifications, since we have $b_1(Y) = 0$ for $d > 1$. Other possible alternatives are Calabi–Yau's, $K3$'s, suitable orbifolds of the type T^d/Z_n, compact hyperbolic manifolds for $d \geq 3$. In general, for spaces having $b_1 \neq 0$ quotienting by an appropriate discrete isometry often leaves us with $b_1 = 0$. This topological classification is insufficient when harmonic spinors are discussed to meet the demand (2). For in that case, the curvature of the manifold (a geometric parameter) plays the decisive role.

This analysis of the massless sector applies to both compactification schemes – tensor or warp.

3.2 Massive bosons: lower bounds

Having considered the massless fields, which are the zero modes of the Laplacian on the internal space, we now turn our attention to the first massive excitations. As we mentioned in the introduction, there has been an extensive study of the first non-zero eigenvalue of the Laplacian on Riemannian manifolds. Rigorous bounds, particularly for manifolds with scalar curvature bounded from below by $(d-1)K$ (where K is constant and d is the dimension of the manifold), have been established. Anyway, assuming a slowly varying κ makes it possible to replace it by $(d-1)K$ in the context of a discussion of mass bounds and scales. It may be noted that, the eigenspectrum being strictly ordered, only the lowest massive states are relevant to our analysis, because if we achieve the decoupling of these, then all the higher modes will automatically be eliminated in the effective four-dimensional theory.

Let Y be a compact manifold, and λ_1 the first non-zero eigenvalue of $\Delta_Y \phi_n = \lambda_n \phi_n$, where ϕ_n is a scalar. Then [12]

$$\lambda_1 + \max\{-(d-1)K, 0\} \geq \frac{\pi^2}{4\sigma^2}, \qquad (3)$$

where σ is the diameter of the manifold. It is worthwhile to note from (3) that the fundamental parameter for masses arising from compactification is σ, and not generically the volume of the manifold, as commonly is thought[5]. However, in certain cases one can proceed a step further and relate σ to the volume of the manifold, and hence rewrite the bounds in terms of the volume instead (e.g. in S^d and certain compact hyperbolic manifolds). The inequality (3) translates effectively into a statement about the bounds on the four-dimensional masses[6] of the lowest excitations, m_1^2. It is obvious that when the Ricci curvature, κ, of Y is non-negative, then one recovers the standard scenario: $\lambda_1 \geq \pi^2/(4\sigma^2)$, where in the standard KK scenario, as in [2], σ is identified with the diameter of the compactification circle(s).

At this point, we note that the explicit expression of the bounds will depend on the nature of the product between the two manifolds – a tensor or a warp product. In the tensor product case, the bound (3) on first scalar excitations (in the four-dimensional effective theory) will remain unaltered,

$$m_1^2 \geq \frac{\pi^2}{4\sigma^2} - \max\{-(d-1)K, 0\}.$$

A natural choice would be $\sigma^{-1} \sim M_*$ (say $\sim \mathcal{O}(\mathrm{TeV})$). In the case of κ bounded from below by a negative constant (i.e. not everywhere positive), the bound will involve the infimum of the curvature (or the curvature itself, if constant or slowly varying), and in order to achieve $m_1^2 \gtrsim \mathrm{TeV}^2$ we need

$$\kappa \approx |(d-1)K| \lesssim \left(\frac{\pi^2}{4} - 1 \right) \mathrm{TeV}^2. \qquad (4)$$

It is remarkable that satisfying this bound on the curvature requires no fine-tuning at all[7]. As was noticed in [8], some manifolds with negative scalar curvature, like compact hyperbolic ones, may have attractive features like exponentially large KK masses. We would like to speculate that negatively curved internal spaces may also be favored (beside the string inspired Ricci flat compactifications), because they support the existence of massless spinors, as will be shown in the next section.

In the warp product case, it is not as easy to comment at this level, mainly because the eigenvalue of Δ_4, $\tilde{m}_1^2$, will be y-dependent in the D-dimensional theory:

$$\tilde{m}_1^2 \geq e^{-\phi(y)} \left[\frac{\pi^2}{4\sigma^2} - \max\{-(d-1)K, 0\} - \frac{1}{2} \left(\partial_l \phi(y) \right) \partial^l \right], \qquad (5)$$

and therefore one cannot interpret $\tilde{m}_1$ as the effective four-dimensional mass. It is not straightforward to decouple the

[4] The number of zero modes of the Laplacian (or equivalently the dimension of the space harmonic p-forms) on a compact manifold Y are given by the pth Betti numbers $b_p(Y)$ of the manifold

[5] This can easily be understood by observing that it is possible to change the spectrum of the Laplacian by deforming the manifold, and yet keep its volume fixed. However, the relation between M_* and M_{P} will always involve the volume of Y

[6] Here, and elsewhere, we use the rest frame when referring to massive states

[7] For large extra dimensions models that have anything to do with string/M-theory, one must have $d \leq 7$

eigenvalue problem for operators in the internal space, because to start with one cannot "eliminate" the warp factor by a rescaling of the fields and still maintain desirable features like square integrability.

It is clear from (2) that both the warp factor and the term $\partial_l \phi(y) \partial^l$ (which should be understood as the gradient of the wave function in the internal space) will change the interpretation of the effective four-dimensional mass. Thus, within this general framework and without any further a priori specifications, one can interpret (5) as a parametric expression and use it in an effective field theory, where integration over all the internal space coordinates would give a final arithmetic expression for the bounds on the masses. This conclusion is in contrast with the bounds on graviton excitations discussed in [8].

Whereas the third demand, of the scalar sector in the theory[8], can be met in the tensor product case (by choosing Y with an appropriate diameter) it seems difficult to be fulfilled without further model-dependent details, in the warp product case.

The same arguments can be carried over to the case of vectors and rank two tensors arising from the gravity sector, as a consequence of the linearization procedure. In principle, one could expect curvature-dependent additions to the equations of motion, as can be read off from (14), but those additional terms are dropped because they involve $\mathcal{O}(h^2)$. In this context, we would like to point out that the vector degrees of freedom, resulting from the metric decomposition, cannot in general be eliminated by a gauge choice, and their amplitudes of coupling leading to a typical scattering are comparable to those of graviton exchange [13]. Hence, it is important to make them very massive and weakly coupled. On the other hand, the curvature-dependent terms in (14) will appear in the YM equations of motion, and it will be difficult to draw conclusions[9], apart from the special case when $d = 2$ where the bounds for the 1-forms are the same as in the case of scalars [14]. In any case, from (14), it can be argued that the bounds for these fields are of the same order as in (3).

3.3 Massive bosons: upper bounds

Finally, we would like to add the following remark. Although the first non-zero eigenvalue of Δ is bounded from below, it is not possible in general to push it to an infinitely heavy scale. There exists an *upper* bound which depends on the same parameter σ. For example, if the Ricci curvature ≥ 0 then the nth eigenvalue ($n \in \mathbf{Z}$) is bounded from above by [15]

$$\lambda_n \leq \frac{2n^2}{\sigma^2} d(d+4). \tag{6}$$

And in the case when the Ricci curvature is bounded from below by a negative number ($K < 0$), then the upper bound [15] becomes more complicated; this includes the (lower bound of) the curvature. For $d \geq 2$ the bound is

$$\lambda_n \leq \frac{(2l-1)^2}{4} K + \frac{4\pi^2 n^2}{\sigma^2} (1 + 2^{l-1})^2, \tag{7}$$

for $d = 2l$, $l = 1, 2, ...$, and

$$\lambda_n \leq l^2 K + \frac{4(1+\pi^2) n^2}{\sigma^2} (1 + 2^{2l-2})^2, \tag{8}$$

for $d = 2l + 1$, $l = 1, 2, ...$ Concerning 1-forms, the same upper bound holds for $\lambda_1^{(1)}$, because $\lambda_1^{(1)} \leq \lambda_1$ [14] (without any further assumption concerning the curvature).

In the tensor product case, the bounds (6), (7) and (8) on the effective four-dimensional masses remain unaltered. However, a more careful treatment, as discussed in Sect. 3.2, is required in the warp product case.

4 Bounds on fermionic KK masses

The Dirac operator on the spin manifold W acting on spinors is $\hat{\not{D}} = \gamma^{\bar{k}}(X) \partial_{\bar{k}}$, where the $\gamma^{\bar{k}}(X)$'s are the D-dimensional gamma matrices in curved space written in terms of the Vielbeins on W. It decomposes in the tensor product case as

$$\hat{\not{D}} = \not{D}_4 + \not{D}_Y, \tag{9}$$

where $\not{D}_Y = \gamma^l(y) \partial_l$, $\not{D}_4 = \gamma^\mu(x) \partial_\mu$; $\gamma^l(y)$, and $\gamma^\mu(x)$ being the gamma matrices of the four- and $(D-4)$-dimensional spaces, respectively (the μ's run over M_4). In the warp product case, the $\not{D}$ splits differently from the one in (9), and has the form

$$\hat{\not{D}} = \mathrm{e}^{(1/2)\phi(y)} \not{D}_4 + \not{D}_Y, \tag{10}$$

where $\not{D}_4$ and $\not{D}_Y$ are the same operators as defined previously. Hence, the four-dimensional fermion masses are related to the eigenvalues of the Dirac operator on the internal manifold[10]. And, in particular, the observed massless fermions in four dimensions are nothing but the zero modes of $\not{D}_Y$ (which lie in $\ker \not{D}_Y$). It has been shown by Lichnerowicz [16] that not all manifolds admit harmonic (massless) spinors. The argument is based on the relation between the squared Dirac operator and the scalar curvature,

$$\not{D}_Y^2 = \nabla^* \nabla + \frac{1}{4} \kappa, \tag{11}$$

where κ is the scalar curvature of Y, ∇^* is the adjoint of ∇, and $\nabla^* \nabla$ is the connection Laplacian (a positive operator). We use this theorem not only to identify candidate manifolds in which the demand (2) can be realized, but also to set geometrical bounds so that (3) is met.

[8] This bound applies on *any* massless scalar field in the theory, whether or not in the gravity sector

[9] Unfortunately, no rigorous bounds for Laplacians on 1-forms, relevant to our discussion, have been worked out

[10] We define the spinor $(\text{mass})^2$ as an eigenvalue of the squared Dirac operator

4.1 Massless fermions

Recall that a spinor ψ is said to be *harmonic* iff $\not{D}\psi = 0$, i.e $\psi \in \ker\not{D}$, It is helpful to remember that $\ker\not{D} = \ker\not{D}^2$, and that this space is finite dimensional, [17], and this space is identified with our space of massless fermions, as mentioned above. It has been shown in [16] that the existence of harmonic spinors depend strongly on the scalar curvature of the manifold, and, in particular, massless spinors do not exist on manifolds with a positive scalar curvature[11]. This no-go theorem applies also to cases where the scalar curvature is non-negative everywhere, and not necessarily constant. In addition, the formula (11) shows that the fermion mass squared is bounded from below by the curvature since the operator $\nabla^*\nabla$ is positive.

According to the above argument, meeting the second demand, namely supporting massless spinors (which will eventually acquire mass only through the Higgs mechanism), rules out the entire class of manifolds with positive curvature, unless they have further discrete isometries. If one wants to relax the second demand, by having the above mass term, then careful attention should be paid so as not to spoil gauge invariance. For instance, a direct mass term in the action for the SM fermions is not gauge invariant. So, adding such a tree-level mass term by hand, and yet keeping gauge invariance, will be at the cost of doubling (or increasing) the number of degrees of freedom (and hence the number of SM generations) depending on the dimension of the spinor representation in the D-dimensional space.

The above price has to be paid anyway, on either negatively or positively curved manifolds, when one goes beyond $D = 6$. It can be shown that $D = 6$ is the maximum dimension for which it is possible to end up with a four-dimensional Weyl spinor (starting from a six-dimensional Weyl one), without extra degress of freedom. Starting from an irreducible spin representation of $SO(1, D-1)$, and after some algebra, the resulting number of four-dimensional Weyl spinors is $(2^{(D-5)/2} \times n)$ for D odd (although special care should be taken in order to define spinors in odd dimensions [17]), and $(2^{D/2-3} \times n)$ for D even, where n is the number of zero modes of the Dirac operator in the internal space (on a compact manifold, the eigenstates are all square integrable). This number n depends on the coupling of spinors to background fields. Hence, the number of the zeroth KK fermionic modes will increase, possibly leading to a variant number of flavors.

A common way to get rid of the above extra spinorial degrees of freedom is to use a localization mechanism as first proposed by [18]. These mechanisms rely on the existence of more than one zero mode of the Dirac operator in the internal space, such that at least one of them is *not* normalizable. Therefore, a necessary condition for applying such scenarios is to have a *non*-compact internal space, because all the modes of a given Dirac operator on a compact space are normalizable. Hence, the recently discussed mechanisms [1,2,19,20] break down for the compact manifold, and extra care is needed for dealing with the additional modes (specially the zero ones). Finally, we mention that the arguments contained here, concerning the zero KK modes, are generic in the sense that they do not depend on whether the product is tensor or warp.

[11] As an example: massless spinors do not exist on a sphere

4.2 Massive fermions: lower bounds

As is the case for the Laplacian, the eigenvalues of the Dirac operator on a compact space are discrete. Therefore, the eigenvalues of the squared Dirac operator are discrete and positive, and in addition *any* eigenvalue, ν_q^2, is bounded from below by the curvature [21], including ν_1^2:

$$\nu_1^2 \geq \frac{d}{4(d-1)}\lambda_1, \tag{12}$$

where λ_1 is the first eigenvalue of the Yamabe operator,

$$L \equiv \frac{4(d-1)}{d-2}\Delta_Y + \kappa,$$

with Δ_Y being the positive Laplacian acting on functions. The implication of the appearance of the Laplacian once again in this fermionic context is that there will be an input from the bosonic spectrum (as transpires from (12) and (3) above) in setting the bound on the massive fermionic excitations. Therefore, the bounds on spin 1/2 and spin 0 masses are not totally independent:

$$\nu_1^2 \geq \frac{d}{d-2}\left[\frac{\pi^2}{4\sigma^2} - \max\{-(d-1)K, 0\}\right] + \kappa.$$

So, for positive curvature ($K > 0$)

$$\nu_1^2 \geq \left(\frac{d}{d-2}\right)\frac{\pi^2}{4\sigma^2} + \kappa,$$

and for negative curvature ($K < 0$)

$$\nu_1^2 \geq \frac{d}{d-2}\left[\frac{\pi^2}{4\sigma^2} + (d-1)K\right] + \kappa.$$

In the tensor product case, the above bounds read the same for the four-dimensional masses, the μ_q's. Thus, by choosing $\sigma^{-1} \sim \mathcal{O}(\text{TeV})$, we find that it is natural to achieve $\mu_1^2 \geq \text{TeV}^2$. In the case $\kappa \geq 0$, all $\mu_q^2 \gtrsim \text{TeV}^2$ without any specific value of the curvature. However, when $\kappa < 0$, the curvature should satisfy an inequality similar to (4):

$$\kappa \approx |(d-1)K| \lesssim \left(\frac{\pi^2 d}{d-2} - 1\right)\text{TeV}^2.$$

It is again remarkable that $\mu_1^2 \gtrsim \text{TeV}^2$ can naturally be achieved having all our mass parameters of the same order as the compactification mass scale. As can be seen from the above inequalities, both σ and the curvature explicitly enter the expressions of the bounds, and hence set the compactification mass scale.

4.3 Massive fermions: upper bounds

Again, as for the Δ eigenmodes, an upper bound on ν_q^2 exists, namely

$$\nu_q^2 \leq Cq^{2/d},$$

where C is a constant that depends only on the geometry of Y (even in the presence of a gauge field) [22]. Again here we find restrictions, though not as explicit as in (6), (7) and (8), which limit our freedom in pushing up the KK masses arbitrarily high.

All the above observations, concerning both the upper and lower bounds, have been done in the tensor product case, though the comments on zero modes apply equally to both types. However, if the product is warp, then the bounds and fermion masses will be dressed by the factor $\mathrm{e}^{-\phi/2}$, as seen from (10), and arguments similar to the ones in Sect. 3.2 apply.

5 Conclusions

We considered, on general grounds, a model of Einstein gravity coupled to a Dirac spinor and a Yang–Mills gauge theory on $W = M_4 \otimes Y$, where Y is a compact internal manifold with a scalar curvature bounded from below, and M_4 is our four-dimensional world. Both the tensor product and the warp product are discussed. Bounds and estimates on the masses of the effective four-dimensional theory at the classical level have been pointed out. Topological restrictions in choosing the internal manifold have been identified in order to avoid having certain bosonic massless modes in the four-dimensional spectrum. In addition, an upper bound on the curvature of Y has been proposed, in the case of non-positive curvature. Geometrical *upper* and *lower* bounds have been presented for both boson and fermion masses. In the tensor product case, the characteristic compactification mass scale for bosons is the diameter of the internal manifold, σ^{-1}, along with $|K|$ when the curvature $\kappa < 0$. For fermions, the compactification mass scale is always set by σ and the curvature, and this is due to an input from the bosonic spectrum in setting the bound on the massive fermionic excitations. Therefore, there is an interplay between the spin 1/2 and spin 0 sectors.

For both fermions and bosons, it turns out that having the masses of the lowest excitations $\gtrsim$ TeV is naturally achieved by taking all the dimensionful parameters, arising from compactification, to be $\sim \mathcal{O}(\text{TeV})$ (no fine-tuning required). In the warp product case, no direct bounds can be applied for massive states without the knowledge of both the specific shape of the warp factor and the field dependence on the internal space, though we are able to implement general estimates. "Zero-mode" arguments can be applied to both kinds of products. From the analysis conducted in this work, we conclude that non-positively curved internal manifolds with $b_0 = 1$ and $b_1 = 0$ are strongly favored for phenomenological purposes.

Finally, a comment about *non-compact* internal manifolds: it has been argued [23] that the spectrum of the Laplacian on non-compact spaces of finite volume has a discrete sector. Moreover, it has been shown recently [24] that a suitable choice of spin structure also leads to a discrete spectrum of the Dirac operator for *non-compact* hyperbolic manifolds of finite volume. One can therefore contemplate analyzing similar bounds for such non-compact spaces, along the same lines as we have done in this work, and discuss their phenomenological implications [25].

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Appendix

The expressions of the Laplacian acting on various tensors have been worked out in [11], and for convenience we list here the relevant expressions:

$$\begin{aligned} \Delta\alpha &= -\nabla_i\nabla^i\alpha = -\frac{1}{\sqrt{g}}\partial(\sqrt{g}g^{ik}\partial_k)\alpha, \\ (\Delta\alpha)_r &= -\nabla_i\nabla^i\alpha_r - \mathcal{R}_r^h\alpha_h, \qquad (13) \\ (\Delta\alpha)_{kl} &= -\nabla_i\nabla^i\alpha_{kl} + \mathcal{R}_k^h\alpha_{hl} + \mathcal{R}_l^h\alpha_{kh} - 2\mathcal{R}_{ki,lh}\alpha^{ih}, \end{aligned}$$

where $i, j, ..., = 1, ...,$ up to the dimension of the manifold on which the tensors and Laplacians are defined. $\mathcal{R}_k^h$ and $\mathcal{R}_{ki,lh}$ are Ricci and Riemann tensors, respectively.

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